Relativistic quantum nonlocality for the three-qubit Greenberger-Horne-Zeilinger state

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(Dated: January 4, 2012)

Lorentz transformation of three-qubit Greenberger-Horne-Zeilinger (GHZ) state is studied. Also we obtain the relativistic spin joint measurement for the transformed state. Using these results it is shown that Bell's inequality is maximally violated for three-qubit GHZ state in relativistic regime. For ultrarelativistic particles we obtain the critical value for boost speed which Bell's inequality is not violated for velocities smaller than this value. We also show that in ultrarelativistic limit Bell's inequality is maximally violated for GHZ state.

PACS numbers: 71.10.Ca ; 41.20.Jb

Relativistic effects on quantum entanglement and quantum information is investigated by many authors [1-13]. Alsing and Milburn [1] studied the Lorentz transformation of maximally entangled states. By explicit calculation of the Wigner rotation they described the observation of the entangled Bell states from two inertial frames moving with the constance velocity with respect to each other. They concluded that entanglement is Lorentz invariant. Terashima, *et al.* [2] considered to relativistic Einstein-Podolsky-Rosen correlation and Bell's inequality. They showed that the degree of the violation of Bell's inequality decreases with increasing the velocity of the observers if the directions of the measurement are fixed. They extended these considerations to the massless case. Ahn, *et al.* [3] investigated the Bell observable for entangled states in the rest frame seen by the moving observer and showed that the entangled states satisfy the Bell's inequality when the boost speed approaches the speed of light. D. Lee, *et al.* [4] showed that maximal violation of the Bell's inequality can be achieved by properly adjusting the directions of the spin measurement even in a relativistically moving inertial frame. Kim, *et al.* [5] obtained an observer-independent Bell's inequality, so that it is maximally violated as long as it is violated maximally in the rest frame. They showed that the Bell observable and Bell states for Bell's inequality should be transformed following the principle of relativistic covariance, which results in a frame independent Bell's inequality.

In this paper we would like to study the Bell's inequality for three-qubit GHZ state in relativistic regime. For doing this, we need the Lorentz transformation of GHZ state and relativistic spin joint measurement.

The following paper is organized as follows. First we review the representation of the Lorentz group and Wigner little group. Then we calculate the Lorentz transformation of three-qubit GHZ state. After that we obtain the relativistic spin joint measurement of GHZ state and calculate the degree of violation for a special case which in non relativistic case gives the maximally violation of Bell's inequality. Finally we calculate the degree of violation for GHZ state when particles moving with same momentum and particles moving in the center of mass frame. Finally we compare our results with two qubit case.

A multipartite state is expressed by

$$\Phi_{\vec{p}_1\sigma_1,\vec{p}_2\sigma_2,...} = a^\dagger(\vec{p}_1,\sigma_1)a^\dagger(\vec{p}_2,\sigma_2)...\Phi_0, \tag{1}$$

where $\vec{p}_i$ is three momentum vector, σ_i is spin label, $a^\dagger$ is creation operator and Φ_0 is Lorentz invariant vacuum state. Multipartite state (1) has the Lorentz transformation property [14]

$$U(\Lambda)\Phi_{\vec{p}_1\sigma_1,\vec{p}_2\sigma_2,...} =$$

$$\sum_{\bar{\sigma}_1\bar{\sigma}_2...} D^{(j_1)}_{\bar{\sigma}_1\sigma_1}\left(W(\Lambda,p_1)\right) D^{(j_2)}_{\bar{\sigma}_2\sigma_2}\left(W(\Lambda,p_2)\right)...\Phi_{\vec{p}_{1\Lambda}\sigma_1,\vec{p}_{2\Lambda}\sigma_2,...}. \tag{2}$$

Here $\vec{p}_{1\Lambda}$ is the three vector part of Λp_1, $D^{(j)}_{\bar{\sigma}\sigma}$ is the unitary spin-j representation of the three dimensional rotation group, and $W(\Lambda,p)$ is Wigner's little group element

$$W(\Lambda,p) = L^{-1}(\Lambda p)\Lambda L(p), \tag{3}$$

where $L(p)$ is the standard boost that takes a particle of mass m from rest to four-momentum p^μ. Transformation of creation operator is

$$U(\Lambda)a^\dagger\left(\vec{p},\sigma\right)U^{-1}(\Lambda) = \sum_{\sigma'} D^{(j)}_{\sigma'\sigma}(W(\Lambda,p))a^\dagger(\vec{p}_\Lambda,\sigma'). \tag{4}$$

The Wigner representation of the Lorentz group for spin-$\frac{1}{2}$ is

$$D(W(\Lambda,p)) = \cos\frac{\delta_{\vec{p}}}{2} + i(\vec{\sigma}\cdot\vec{n})\sin\frac{\delta_{\vec{p}}}{2} = \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix}, \tag{5}$$

with

$$\cot\frac{\delta_{\vec{p}}}{2} = \coth\frac{\xi}{2}\coth\frac{\chi}{2} + \hat{e}\cdot\hat{p}, \tag{6}$$

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where

$$\cosh\chi = (p^0/m), \qquad \tanh\xi = \beta = v/c, \qquad \vec{n} = \hat{e}\times\hat{p}. \tag{7}$$

Here $\hat{e}$ is a normal vector in the boost direction and v is the boost speed. We consider the case in which the boost speed is perpendicular to momentums of particles. In this case we have

$$\cos\frac{\delta}{2} = \left[\frac{(1+\sqrt{1-\beta^2})(\cosh\chi+1)}{2(\sqrt{1-\beta^2}+\cosh\chi)}\right]^{1/2}, \tag{8}$$

$$\sin\frac{\delta}{2} = \left[\frac{(1-\sqrt{1-\beta^2})(\cosh\chi-1)}{2(\sqrt{1-\beta^2}+\cosh\chi)}\right]^{1/2}, \tag{9}$$

where in ultrarelativistic limit as $\beta\to 1$ take the forms

$$\cos\frac{\delta}{2} \to \left[\frac{1+\mathrm{sech}\chi}{2}\right]^{1/2}, \tag{10}$$

$$\sin\frac{\delta}{2} \to \left[\frac{1-\mathrm{sech}\chi}{2}\right]^{1/2}. \tag{11}$$

Investigations show that exist a family of pure entangled $N>2$ qubit states that do not violate any Bell's inequality for N-particle correlations for the case of a standard Bell experiment on N qubits [15]. For $N=3$, one class is Greenberger-Horne-Zeilinger (GHZ) state given by $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle+|111\rangle)$, the other class is represented by the W state $|W\rangle = \frac{1}{\sqrt{3}}(|110\rangle+|101\rangle+|011\rangle)$, where 0 and 1 represent spins polarized up and down along the z axis.

We can express GHZ state using creation operator in the rest frame

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left\{a^\dagger(\vec{p}_1,0)a^\dagger(\vec{p}_2,0)a^\dagger(\vec{p}_3,0)\right.$$

$$\left.+a^\dagger(\vec{p}_1,1)a^\dagger(\vec{p}_2,1)a^\dagger(\vec{p}_3,1)\right\}\Phi_0. \tag{12}$$

For simplicity we assume that momentum of particles are sufficiently localized around momentum $\vec{p}_i$. Realistic situation involve the wave pockets with gaussian form $\exp(-\vec{p}_i^2/2\Delta^2)$ with characteristic spread Δ. Note that these particles are indistinguishable. Authors in reference [12] investigated that one can create distinguishable qubits from indistinguishable particles by preparing particles in minimum uncertainty states that are well localized with a sharp momentum. They show that N-qubit product state can be constructed from N single particle states as

$$|\psi\rangle_N = \otimes_{n=1}^N e^{-iaP_z}|\psi\rangle_1, \tag{13}$$

where $|\psi\rangle_1$ is a single particle state. State (13) describes a one dimensional latices of particles with separation a. Using a proton (hydrogen atom) in the millikelvin range as an example, condition for distinguishablity is $a \gg 100A^\circ$.

Using relation (4) Lorentz transformation of GHZ state becomes

$$|GHZ'\rangle = \frac{1}{\sqrt{2}}(A|000\rangle + B|001\rangle + C|010\rangle + D|011\rangle$$

$$+E|100\rangle + F|101\rangle + G|110\rangle + H|111\rangle)|\vec{p}_1\vec{p}_2\vec{p}_3\rangle_\Lambda, \tag{14}$$

with

$$A = D^1_{00}D^2_{00}D^3_{00} + D^1_{01}D^2_{01}D^3_{01},$$

$$B = D^1_{00}D^2_{00}D^3_{10} + D^1_{01}D^2_{01}D^3_{11},$$

$$C = D^1_{00}D^2_{10}D^3_{00} + D^1_{01}D^2_{11}D^3_{01},$$

$$D = D^1_{00}D^2_{10}D^3_{10} + D^1_{01}D^2_{11}D^3_{11},$$

$$E = D^1_{10}D^2_{00}D^3_{00} + D^1_{11}D^2_{01}D^3_{01}$$

$$F = D^1_{10}D^2_{00}D^3_{10} + D^1_{11}D^2_{01}D^3_{11},$$

$$G = D^1_{10}D^2_{10}D^3_{00} + D^1_{11}D^2_{11}D^3_{01},$$

$$H = D^1_{10}D^2_{10}D^3_{10} + D^1_{11}D^2_{11}D^3_{11}, \tag{15}$$

where D^i is Wigner representation for particle i. The generalization of the Bell's type inequality to the case of three particles is the one proposed by Mermin which can be expressed in terms of correlation functions as follows [16]

$$\varepsilon = |E(\vec{a},\vec{b},\vec{c'}) + E(\vec{a},\vec{b'},\vec{c}) + E(\vec{a'},\vec{b},\vec{c}) - E(\vec{a'},\vec{b'},\vec{c'})| \le 2, \tag{16}$$

where

$$E(\vec{a},\vec{b},\vec{c}) = \langle\psi|(\vec{a}\cdot\vec{\sigma})\otimes(\vec{b}\cdot\vec{\sigma})\otimes(\vec{c}\cdot\vec{\sigma})|\psi\rangle,$$

is correlator function, $\vec{a}$, $\vec{b}$ and $\vec{c}$ are real three-dimensional vectors of unit length and $\vec{\sigma} = (\sigma_x,\sigma_y,\sigma_z)$ is the Pauli spin operator. For each measurement, one of two possible alternative measurement is performed: $\vec{a}$ or $\vec{a'}$ for particle 1, $\vec{b}$ or $\vec{b'}$ for particle 2, $\vec{c}$ or $\vec{c'}$ for particle 3. For GHZ state, Bell's inequality is violated if, for example, measurements are made in the xy plane along some appropriate directions. In this case

$$E(\vec{a},\vec{b},\vec{c}) = \cos(\phi_1+\phi_2+\phi_3), \tag{17}$$

where we labelled the angles from the x-axis. The correlation function $E(\vec{a},\vec{b},\vec{c})$ can take the value either $+1$ or

-1 under both realistic theory and quantum mechanical theory, thus the maximum value of ε is 4.

The normalized relativistic spin observable $\hat{a}$ is given by [17]

$$\hat{a} = \frac{(\sqrt{1-\beta^2}\vec{a}_\perp + \vec{a}_\parallel)\cdot\vec{\sigma}}{\sqrt{1+\beta^2[(\vec{e}\cdot\vec{a})^2-1]}}, \tag{18}$$

where the subscripts $\perp$ and $\parallel$ denote the components which are perpendicular and parallel to the boost direction. Operator $\hat{a}$ is related to the Pauli-Lubanski pseudo vector which is relativistic invariant operator corresponding to spin. Now we are ready to calculate the relativistic Bell's inequality for three particles system. Spin joint measurement for the transformed state $|GHZ'\rangle$ for measurement in xy plane is

$$\langle GHZ'|\hat{a}\otimes\hat{b}\otimes\hat{c}|GHZ'\rangle$$

$$= \left\{[1+\beta^2(a_x^2-1)][1+\beta^2(b_x^2-1)][1+\beta^2(c_x^2-1)]\right\}^{-1/2}$$

$$\times\Re(E^*Da_{xy}b_{xy}^*c_{xy}^* + F^*Ca_{xy}b_{xy}^*c_{xy}$$

$$+G^*Ba_{xy}b_{xy}c_{xy}^* + H^*Aa_{xy}b_{xy}c_{xy}), \tag{19}$$

where $a_{xy} = a_x + ia_y\sqrt{1-\beta^2}$ and so on. In ultra relativistic limit as $\beta \to 1$ we get

$$\langle GHZ'|\hat{a}\otimes\hat{b}\otimes\hat{c}|GHZ'\rangle$$

$$\to \frac{a_xb_xc_x}{|a_xb_xc_x|}\Re\{AH^* + G^*B + F^*C + E^*D\}, \tag{20}$$

which is not correlated. In non-relativistic limit

$$\langle GHZ'|\hat{a}\otimes\hat{b}\otimes\hat{c}|GHZ'\rangle$$

$$\to a_xb_xc_x - a_yb_xc_y - a_yb_yc_x - a_xb_yc_y$$

$$= \cos(\phi_1+\phi_2+\phi_3). \tag{21}$$

Here we consider to the vector set inducing the maximal violation of Bell's inequality for GHZ state in non relativistic case. With the following suitably chosen measurement settings,

$$\vec{a} = \vec{b} = \vec{c} = \hat{y},$$

$$\vec{a'} = \vec{b'} = \vec{c'} = \hat{x}, \tag{22}$$

and using the algebra of pauli matrices we have

$$(\sigma_x\sigma_x\sigma_x - \sigma_y\sigma_y\sigma_x - \sigma_y\sigma_x\sigma_y - \sigma_x\sigma_y\sigma_y)|GHZ\rangle = 4|GHZ\rangle, \tag{23}$$

then for GHZ state Bell's inequality is maximally violated with $|\varepsilon| = 4$. For set vectors (22) the relativistic Bell measurement becomes

$$\varepsilon' = 4\Re\{AH^*\}. \tag{24}$$

We obtain the degree of violation for two cases.

Cass I. $\vec{p_1} = \vec{p_2} = \vec{p_3} = p\hat{z}$

In this case

$$D(W(\Lambda,p_1)) = D(W(\Lambda,p_2)) = D(W(\Lambda,p_3))$$

$$= \begin{pmatrix} \cos\frac{\delta}{2} & -\sin\frac{\delta}{2} \\ \sin\frac{\delta}{2} & \cos\frac{\delta}{2} \end{pmatrix}, \tag{25}$$

and Bell observable takes the form

$$\varepsilon' = \cos^3\delta + 3\cos\delta. \tag{26}$$

In ultrarelativistic limit as $\beta \to 1$, (26) reduces to

$$\varepsilon' \to \text{sech}^3\chi + 3\text{sech}\chi \leq 4. \tag{27}$$

In this limit amount of violation for very high energy particles goes to zero, but for low energy particles approaches to 4, similar to nonrelativistic limit $\beta \to 0$.

Cass II. $\vec{p_1} + \vec{p_2} + \vec{p_3} = 0$

The particles are in the center of mass frame with the following momentums

$$\vec{p_1} = -p\hat{z},$$

$$\vec{p_2} = \left(\frac{1}{2}\hat{z} + \frac{\sqrt{3}}{2}\hat{y}\right)p,$$

$$\vec{p_3} = \left(\frac{1}{2}\hat{z} - \frac{\sqrt{3}}{2}\hat{y}\right)p. \tag{28}$$

Wigner representations of the the Lorentz group for particles 1, 2 and 3 respectively are written as

$$D(W(\Lambda,p_1)) = \begin{pmatrix} \cos\frac{\delta}{2} & \sin\frac{\delta}{2} \\ -\sin\frac{\delta}{2} & \cos\frac{\delta}{2} \end{pmatrix}, \tag{29}$$

$$D(W(\Lambda,p_2)) = D^*(W(\Lambda,p_3))$$

$$= \begin{pmatrix} \cos\frac{\delta}{2} + i\frac{\sqrt{3}}{2}\sin\frac{\delta}{2} & -\frac{1}{2}\sin\frac{\delta}{2} \\ \frac{1}{2}\sin\frac{\delta}{2} & \cos\frac{\delta}{2} - i\frac{\sqrt{3}}{2}\sin\frac{\delta}{2} \end{pmatrix}. \tag{30}$$

In this case Bell observable to be

$$\varepsilon' = \frac{1}{16}\cos^3\delta + \frac{3}{8}\cos^2\delta + \frac{33}{16}\cos\delta + \frac{3}{2}. \tag{31}$$

which for ultrarelativistic limit as $\beta \to 1$ reduces to

$$\varepsilon' \to \frac{1}{16}\text{sech}^3\chi + \frac{3}{8}\text{sech}^2\chi + \frac{33}{16}\text{sech}\chi + \frac{3}{2}. \tag{32}$$

For very high energy particles amount of violation is $\varepsilon' = 1.5$, but for low energy particles $\varepsilon' = 4$ which is maximally violation of Bell's inequality.

From the two preceding cases it is obvious that, the degree of violation decreases under Lorentz transformation. This is because Bell observable is evaluated with the same spin measurement directions as in the non-relativistic lab frame. By finding a new set of spin measurement directions, for example by rotating the spin measurement directions with Wigner rotation, Bell's inequality is still maximally violated in a Lorentz-boosted frame [2, 4, 5].

It's interesting to express Bell observable in order function of β for ultrarelativistic particles. In this situation relations (8) and (9) reduce to

$$\cos\frac{\delta}{2} \approx \left[\frac{(1+\sqrt{1-\beta^2})}{2}\right]^{1/2}, \tag{33}$$

$$\sin\frac{\delta}{2} \approx \left[\frac{(1-\sqrt{1-\beta^2})}{2}\right]^{1/2}, \tag{34}$$

then the amount of violation (26) takes the form

$$\varepsilon' \approx \sqrt{1-\beta^2}(4-\beta^2). \tag{35}$$

It's obvious that critical value β_c for satisfying Bell's inequality is 0.8. Critical value for case II is 0.97.

Now we compare our results with two-qubit case obtained by Ahn, *et al* [3]. They calculated relativistic Bell observable for two qubit entangled Bell state, when particles move in the center of mass frame, and found

$$\varepsilon' = \frac{2}{\sqrt{2-\beta^2}}(\sqrt{1-\beta^2} + \cos 2\delta). \tag{36}$$

In ultrarelativistic limit $\beta \to 1$: $\varepsilon' \to |4\mathrm{sech}^2\chi - 2| \leq 2$ which indicates the Bell's inequality is not violated in this limit. This result is not same as three-qubit case. For very high energy particles (36) reduces to

$$\varepsilon' \approx \frac{2}{\sqrt{2-\beta^2}}(1+\sqrt{1-\beta^2} - 2\beta^2), \tag{37}$$

the critical value for violation of Bell's inequality in this case is $\beta_c = 0.86$, which is smaller than three-qubit case when particles move in the center of mass frame.

In conclusion using Bell's inequality, we studied the nonlocal quantum properties of GHZ state in relativistic formalism. First we obtained the relativistic spin joint measurements for Lorentz transformed three-qubit GHZ state. We show that in ultrarelativistic limit joint measurement is uncorrelated. We also investigated the degree of violation for particles moving with same momentum and particles moving in the center of mass frame. Bell's inequality is maximally violated in rest frame or in moving frame with rest particles, but as seen by moving observer is not always violated, because the degree of violation of Bell's inequality depends on the velocity of the particles and observer. In non relativistic case the spin degrees of freedom and momentum degrees of freedom are independent. But in relativistic regime Lorentz transformation of spin of particle depends on its momentum. For GHZ state we show that in ultrarelativistic limit Bell's inequality is maximally violated which is not same as two-qubit case. Finally, for very high energy particles we obtained a critical value for satisfying Bell's inequality. The critical value for three-qubit state is greater than two-qubit case.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor E. Solano for his valuable suggestions.

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